An Implementation of Homomorphic Modulo Theory in Yager's Fermatean Fuzzy Structure

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Abstract

In this article, we introduce the notion of a fermatean fuzzy sub-module, extending the theoretical framework of fuzzy algebraic structures. We rigorously prove various characteristics of fermatean fuzzy sub-modules, exploring their unique properties and behavior under algebraic operations. Furthermore, we provide a detailed analysis of the relationship between a fermatean fuzzy small module and a basic small module, highlighting key similarities and differences that contribute to a deeper understanding of module theory in the fuzzy context. Our study also investigates several important properties associated with fermatean fuzzy small modules, shedding light on their structural significance and potential applications. Through these contributions, the article aims to enrich the existing body of knowledge in fuzzy algebra and inspire further research into the properties and uses of fermatean fuzzy modules in both theoretical and applied mathematics. The results presented may open new directions for exploring advanced fuzzy systems and their practical implications.

Keywords: fuzzy set, intuitionistic fuzzy set, pythagorean fuzzy set, fermatean fuzzy set, sub-module, homomorphism

I. INTRODUCTION

Zadeh [17]introduced the concept of fuzzy set which was generalization of the classical set in 1965. This encourages many researchers to investigate set theory in fuzzy setting. Pythagorean fuzzy set is one of the most important fuzzy sets. It's importance ties behind the fact that this set can be applied in order to characterized uncertain data accurately. In order to tackle confusion and unpredictability in decision-making processes more effectively, Senapati.T [11] introduced fermatean fuzzy sets in 2020, an extension of intuitionistic fuzzy set (IFSs) and pythagorean fuzzy set (PFSs). AVs and NAVs of fermatean fuzzy sets reveal their dependence on greater powers with sum of cubes is less than 1. Ibrahim.H.Z introduced and applied n, m-rung orthopair fuzzy sets in MCDM.. The concepts of fuzzy modules and fuzzy sub-modules was introduced by Negoita and Ralescu[10] in 1975. Since then, several authors have studied fuzzy modules[3,7,10]. The concept of essential fuzzy modules was introduced by Hadi [6] in 2000. Using this idea. Abbas established the concept of essential fuzzy sub-modules and uniform fuzzy modules in 2012. In this article**,** we introduce the notion of a fermatean fuzzy sub-module. We prove various characteristics of fermatean fuzzy sub-module. We provide a relation between a fermatean fuzzy small module and a basic small module. Finally, some important properties regarding fermatean fuzzy small modules are investigated.

II. PRELIMINARIES

Definition-2.1: A fuzzy set 'A' in X is a set of ordered pairs $A = \{(x, J_A(x)/ x \in X)\}$, where J_A is the grade of membership of $x \in A$ and $J_A : X \to [0,1]$ is the membership function.

Definition-2.2: Let $A = \{(x, J_A(x)/ x \in X)\}$ be a fuzzy set. The complement of A is defined as $A' = \{(x, K_A(x)/ x \in X)\} = \{(x, K_A(x) = 1 - J_A(x))\}$.

Definition-2.3: A Pythagorean fuzzy set (PFS) 'A' of universe of discourse X is of the form $A = \{(x, J_A(x), K_A(x)/ x \in X)\}$, where $J_A(x)$ and $K_A(x)$ are the membership and non-membership of x respectively in which $0 \le J_A(x) \le 1$, $0 \le K_A(x) \le 1$ and $0 \le {J^2}_A(x) + {K^2}_A(x) \le 1$ for every $x \in X$.

Suppose if the condition $0 \le {J^3}_A(x) + {K^3}_A(x) \le 1$ for every $x \in X$ is called fermatean fuzzy set.

Definition-2.4: Let A, B be fermatean fuzzy sets in a fixed set X. Then

(i) A is a subset of B if for all $x \in X$, we have ${J^3}_A(x) \le {J^3}_B(x)$ and ${K^3}_A(x) \ge {K^3}_B(x)$.

(ii) ${J^3}_{A\cap B}(x) = \min\{{J^3}_A(x), {J^3}_B(x)\}$, for every $x \in X$ and ${K^3}_{A\cap B}(x) = \max\{{K^3}_A(x), {K^3}_B(x)\}$.

(iii) ${J^3}_{A\cup B}(x) = \max\{{J^3}_A(x), {J^3}_B(x)\}$, for every $x \in X$ and ${K^3}_{A\cup B}(x) = \min\{{K^3}_A(x), {K^3}_B(x)\}$.

(iv) ${J^3}_{A+B}(x) = {J^3}_A(x) + {J^3}_B(x) - {J^3}_A(x)\, {J^3}_B(x)$ and ${K^3}_{A+B}(x) = {K^3}_A(x).{K^3}_B(x)$.

Now, we are able to introduce the definition of the fermatean fuzzy sub-module

Definition-2.5: Let M be an R-module and 'A' is called a fermatean fuzzy subset of M. Then 'A' is called a fermatean fuzzy sub-module of M, denoted by $A \le_{FF} M$, if the following conditions are satisfied:

(i) ${J^3}_A(0) = 1$ and ${J^3}_A(1) = 0$.

(ii) ${J^3}_A(x+y) \ge \min\{{J^3}_A(x), {J^3}_A(y)\}$, for all $x, y \in M$ and ${K^3}_A(x+y) \le \max\{{K^3}_A(x), {K^3}_A(y)\}$, for all $x, y \in M$.

(iii) ${J^3}_A(r\,x) \ge {J^3}_A(x)$ and ${K^3}_A(r\,x) \le {K^2}_B(x)$, for all $x \in M$ and $r \in R$.

Recall that for a module M, we define the fermatean fuzzy set ${\chi_M}^{FF} = (\chi_M, {\chi_M}^C)$ in which

$$\chi_M(x) = \begin{cases} 1, & \text{if } x \in M \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad {\chi_M}^C(x) = \begin{cases} 0, & \text{if } x \in M \\ 1, & \text{otherwise} \end{cases}.$$

Definition-2.6: Let M be a module and A be a fermatean fuzzy subset of M. Then

(i) $A^* = J^*{}_A \cap \tilde{J}^*{}_A$, where
$J^*{}_A = \{x \in M / J_A(x) > 0\}$
$\tilde{J}^*{}_A = \{x \in M / J_A(x) < 1\}$

(ii) $A_* = J_{*A} \cap \tilde{J}_{*A}$, where
$J_{*A} = \{x \in M / J_A(x) = 1\}$
$\tilde{J}_{*A} = \{x \in M / J_A(x) = 0\}$

III. STRUCTURES OF FERMATEAN FUZZY SUB-MODULE

Recall that a sub-module N of a module M is called a small subspace of module of M, denoted by $N \ll M$ if $N + S \neq M$ for every proper sub-module S of a module M.
Clearly, the zero sub-module is a small sub-module of any module M.
Moreover, a small sub-module of a module M should be a proper sub-module.

Now, we present some well-known structures(properties) regarding the concept of small sub-modules.

Theorem-3.1: Suppose that M is a module and S, T, N are sub-modules of M such that $S < T$. Then,

(i) $S + N \ll M$ if and only if $S \ll M$ and $N \ll M$.

(ii) $T \ll M$ if and only if $S \ll M$ and $\frac{T}{S} \ll \frac{M}{S}$.

(iii) If $S \ll T$, then $S \ll M$.

Now, we are ready to introduce the main concept in this article.
Consider a module M. Then a fermatean fuzzy set, $A = \big(J_A(x), K_A(x)\big)$ is called a fermatean fuzzy small sub-module of M, denote $A \leq_{FF} M$, if $A + S \neq \chi_M{}^{FF}$ for any $S \neq \chi_M{}^{FF}$. That is whenever $P + S = \chi_M{}^{FF}$, then $S = \chi_M{}^{FF}$.

Theorem-3.2: Let M be a module and A be a sub-module of M. Then $A \ll M$ if $\chi_M{}^{FF} \leq_{FF} M$.
Proof: Suppose that $\chi_A{}^{FF} \leq_{FF} M$ and $P + S = M$ for some proper sub-module S of M. Then for any $a \in M$, there exist $x \in A$ and $y \in S$ such that $x + y = m$. We obtain

$$J^3{}_{\chi_A}{}^{FF} + \chi^3{}_A{}^{FF}(a) = \chi^3{}_A(a) + \chi^3{}_S(a) - \chi^3{}_A(a)\,\chi^3{}_S(a)$$
$$\geq \min\{\chi^3{}_A(x) + \chi^3{}_S(x) - \chi^3{}_A(x)\,\chi^3{}_S(x),\ \chi^3{}_A(y) + \chi^3{}_S(y) - \chi^3{}_A(y)\,\chi^3{}_S(y)\}$$

This means that,

$$J^3{}_{\chi_A}{}^{FF} + \chi_A{}^{FF}(a) = \chi^{C^3}{}_A(a)\,\chi^{C^3}{}_S(a) \leq \max\{\chi^{C^3}{}_A(x)\,\chi^{C^3}{}_S(x), \chi^{C^3}{}_A(y)\,\chi^{C^3}{}_S(y)\} = 0.$$

This means that, $K^3{}_{\chi_A}{}^{FF} + \chi_A{}^{FF} = \chi^{C^3}{}_M$.
Thus $\chi_A{}^{FF} + \chi_S{}^{FF} = \chi_M{}^{FF}$, but this contradicts the facts that $\chi_A{}^{FF} \leq_{FF} M$ and $\chi_S{}^{FF} \neq \chi_M{}^{FF}$ as S is a proper sub-module of M.
∴ A is a small sub-module of M.

Theorem-3.3: Let M be a module and A be a fermatean fuzzy sub-module of M. If $A \leq_{FF} M$ then $A_* \ll M$.
Proof: Assume that $A \leq_{FF} M$. In order to see that $A_* \ll M$, suppose that $A_* + S = M$ for a sub-module S of M.

We aim to sow that $A + \chi_S{}^{FF} = \chi_M{}^{FF}$.

Let $a \in M$. Then $a = x + y$, for some $x \in A$ and $y \in S$. Then
$J^3{}_{A+\chi_S{}^{FF}}(a) = J^3{}_A(a) + \chi^3{}_S(a) - J^3{}_A(a)\,\chi^3{}_S(a)$
$\geq \min\{J^3{}_A(x) + \chi^3{}_S(x) - J^3{}_A(x)\,\chi^3{}_S(x),\ J^3{}_A(y) + \chi^3{}_S(y) - J^3{}_A(y)\,\chi^3{}_S(y)\}$
Moreover,
$K^3{}_{A+\chi_S{}^{FF}}(a) = K^3{}_A(a)\,\chi^{C^3}{}_S(a) \leq \max\{K^3{}_A(x)\,\chi^{C^3}{}_S(x),\ K^3{}_A(y)\,\chi^{C^3}{}_S(y)\}$
Thus $A + \chi_S{}^{FF} = \chi_M{}^{FF}$.
By hypothesis, $\chi_S{}^{FF} = \chi_M{}^{FF}$. Therefore $S = M$.
Example-3.4: Consider the Z –module Z_{20} and the sub-module $S = \langle 5 \rangle$.
Let A be a fermatean fuzzy sub-module of Z_{20} defined as follows.
$J_A(a) = \begin{cases} 1, & \text{if } a \in S \\ 1/4, & \text{otherwise} \end{cases}$ and $K_A(a) = \begin{cases} 1, & \text{if } a \in S \\ 1/6, & \text{otherwise} \end{cases}$
It is clear that A_* is not a small sub-module of Z_{20} as $A_* + \langle 2 \rangle = Z_{20}$.
Thus A is not a fermatean fuzzy small sub-module of Z_{20}.

Corollary-3.5: Let A, S be two fermatean fuzzy sub-module of a module M in which $A \subseteq S$. Then $A \leq_{FF} S$ if and only if $A_* \leq_{FF} S^*$.
Proof: It is obvious.

Theorem-3.6: Let M be a module and S be a sub-module of M and A is a fermatean fuzzy sub-module of a module M in which $A \subseteq \chi_S{}^{FF}$. If $A/_S$ is a fermatean fuzzy small sub-module of S, then A is a fermatean fuzzy small sub-module of M.
Proof: Assume that Q is a fermatean fuzzy sub-module of M such that $A + Q = \chi_M{}^{FF}$.
In order to see that $A/_S + \left(Q/_S \cap \chi_S{}^{FF}\right)$, let $x \in S$. Then we obtain
$J^3{}_{A/_S+\left(Q/_S \cap \chi_S{}^{FF}\right)}(x) = J^3{}_{A/_S}(x) + J^3{}_{Q/_S \cap \chi_S{}^{FF}}(x) - J^3{}_{A/_S}(x) J^3{}_{Q/_S \cap \chi_S{}^{FF}}(x)$
$= J^3{}_{A/_S}(x) + \min\left\{J^3{}_{Q/_S}(x),\ \chi^3{}_S(x)\right\} - J^3{}_{A/_S}(x) \min\left\{J^3{}_{Q/_S}(x),\ \chi^3{}_S(x)\right\}$
$= \min\{J^3{}_A(x),\ \chi^3{}_S(x)\} + \min\left\{J^3{}_Q(x),\ \chi^3{}_S(x)\right\} - \min\{J^3{}_A(x),\ \chi^3{}_S(x)\} \min\left\{J^3{}_Q(x),\right.$
$\left.\chi^3{}_S(x)\right\}$
$= J^3{}_A(x) + J^3{}_Q(x) - J^3{}_A(x) J^3{}_Q(x) = J^3{}_{A+Q}(x) = \chi^3{}_M(x) = 1 = \chi^3{}_S(x)$ and
$K^3{}_{A/_S+\left(Q/_S \cap \chi_S{}^{FF}\right)}(x) = K^3{}_{A/_S}(x)\, K^3{}_{Q/_S \cap \chi_S{}^{FF}}(x)$
$= K^3{}_{A/_S}(x) \max\left\{K^3{}_{Q/_S}(x),\ \chi^{C^3}{}_S(x)\right\}$
$= \max\{K^3{}_A(x),\ \chi^{C^3}{}_S FF(x)\} \max\{K^3{}_Q(x),\ \chi^{C^3}{}_S FF(x)\}$
$= K^3{}_A(x) K^3{}_Q(x) = K^3{}_{A+Q}(x) = \chi^{C^3}{}_M(x) = 0 = \chi^{C^3}{}_S(x)$
This implies that $A/_S + \left(Q/_S \cap \chi_S{}^{FF}\right) = \chi_S{}^{FF}$.
By hypothesis, we conclude that $Q/_S \cap \chi_S{}^{FF} = \chi_S{}^{FF}$.
Thus $\chi_S{}^{FF} \subseteq Q/_S$. Then $\chi_M{}^{FF} = A + T \subseteq T \subseteq \chi_M{}^{FF}$.
Therefore, $Q = \chi_M{}^{FF}$ and A is fermatean fuzzy small sub-module of M.
As a consequence of the above theorem, we have the following corollary.

Corollary-3.7: Let M be a module. Also A and S are fermatean fuzzy sub-modules of M in which A⊆S. If A is fermatean fuzzy small sub-module of S, then A is s fermatean fuzzy small sub-module of M.
Proof: It is obvious.
Note-3.8: The converse of the above theorem need not be true in general. That is if M is a module, S be a sub-module of M and A is a fermatean fuzzy sub-module of M in which $A \subseteq {\chi_S}^{FF}$, then it is not true in general that $A/_S$ is a fermatean fuzzy small sub-module of S. For example take $A/_S = S$.
Proposition-3.9: Let M be a module and A, S, Q be fermatean fuzzy sub-modules of M. Then $(A \cap S) + (A \cap Q) \subseteq A \cap (S \cap Q)$.
Proof: Let $a \in M$. Then

$$J^3{}_{(A\cap S)+(A\cap Q)}(a) = J^3{}_{(A\cap S)}(a) + J^3{}_{(A\cap Q)}(a) - J^3{}_{(A\cap S)}(a)J^3{}_{(A\cap Q)}(a)$$
$$=\min\{J^3{}_A(a),\ J^3{}_S(a)\} + \min\left\{J^3{}_A(a),\ J^3{}_Q(a)\right\} - \min\{J^3{}_A(a),\ J^3{}_S(a)\}\min\left\{J^3{}_A(a),\ J^3{}_Q(a)\right\}$$
$$\leq\min\left\{J^3{}_A(a),\ J^3{}_S(a) + J^3{}_Q(a) - J^3{}_S(a),\ J^3{}_Q(a)\right\} =\min\left\{J^3{}_A(a),\ J^3{}_{S+Q}(a)\right\}$$
$$= J^3{}_{A\cap(S+Q)}(a)$$

Moreover,

$$K^3{}_{(A\cap S)+(A\cap Q)}(a) = K^3{}_{(A\cap S)}(a)K^3{}_{(A\cap Q)}(a) =\max\{K^3{}_A(a), K^3{}_S(a)\}\max\left\{K^3{}_A(a), K^3{}_Q(a)\right\}$$
$$\geq \max\left\{K^3{}_A(a), K^3{}_S(a), K^3{}_Q(a)\right\} = K^3{}_{A\cap(S+Q)}(a)$$

Proposition-3.10: Let M be a module and A and S are fermatean fuzzy sub-modules of M, in which ${\chi_M}^{FF} = A \oplus_{FF} S$. Then $M = A^* \oplus S^* = A_* \oplus S_*$.
Proof: Let $a \in M$. Then

$$1 = \chi^3{}_M(a) = J^3{}_{A+S}(a)$$
$$= J^3{}_A(a) + J^3{}_S(a) - J^3{}_A(a)J^3{}_S(a)$$
$$= J^3{}_A(a)\left(1 - J^3{}_S(a)\right) + J^3{}_S(a).$$

This implies that, $J^3{}_A(a) = 1$ or $J^3{}_S(a) = 1$. So that $J^3{}_A(a) = 0$ or $J^3{}_S(a) = 0$.
Hence $a \in A_*$ or $a \in S_*$, so that $M = A + S_*$.
Hence $M = A^* + S^*$.
Our aim is to show that $A^* \cap S^*$.
Assume that $a \in A^* \cap S^*$. Then $J^3{}_P(a)J^3{}_S(a) > 0$.
Since ${\chi_M}^{FF} = A \oplus_{FF} S$, we obtain $0 < \min\{J^3{}_P(a),\ J^3{}_S(a)\} = \chi^3{}_0(a)$.
Which means that $a = 0$ and hence, $A_* \cap S_* \subseteq A^* \cap S^* = 0$.
Hence the proof.

Now, we able to clear that the converse of corollary-3.7 is true if S is a fermatean fuzzy direct summand of M as follow.

Proposition-3.11: Let M be a module and A and S are fermatean fuzzy sub-modules of M, in which A⊆S and S is a fermatean fuzzy direct summand of M. Then A is fermatean fuzzy small sub-module of S if and only if A is fermatean fuzzy sub-module of M.
Proof: Assume that A is a fermatean fuzzy small sub-module of M.

Applying theorem-3.3, A_* is a small sub-module of M. That S is a fermatean fuzzy direct summand of M and $A_* \subseteq S^*$, implies that A_* is a small sub-module of S^*.

Applying corollary-3.7, this result hold.

Theorem-3.12: Let M be a module and S be fermatean fuzzy sub-module of M such that $A \cap S = \chi_0{}^{FF}$. Then

(i) $(A \oplus_{FF} S)^* = A^* \oplus S^*$,

(ii) $(A \oplus_{FF} S)_* = A_* \oplus S_*$.

Proof:

(i) Since $A \cap S = \chi_0{}^{FF}$, we need to show that $(A + S)^* = A^* + S^*$.
Suppose that $a \in (A + S)^*$. By definition, we have $J^3{}_{A+S}(a) > 0$.
This implies that $0 < J^3{}_A(a) + J^3{}_S(a) - J^3{}_A(a)J^3{}_S(a)$
$= J^3{}_A(a)\left(1 - J^3{}_S(a)\right) + J^3{}_S(a)$
This means that $J^3{}_A(a) \neq 0$ or $J^3{}_S(a) \neq 0$.
Moreover, $1 > K^3{}_{A+S}(a) = K^3{}_A(a)K^3{}_S(a)$.
This implies that $K^3{}_A(a) < 1$ or $K^3{}_S(a) < 1$.
Thus $a \in A^*$ or $a \in S^*$, so that $a \in A^* + S^*$ and $(A + S)^* \subseteq A^* + S^*$.
Now, suppose that $a = x_1 + y_1 \in A^* + S^*$, where $x_1 \in A^*$ and $y_1 \in S^*$.
By definition, $J^3{}_A(x_1), J^3{}_S(y_1) > 0$. Thus,

$$0 < \min\left\{J^3{}_A(x_1)\left(1 - J^3{}_S(x_1) + J^3{}_S(x_1)\right),\ J^3{}_A(y_1)\left(1 - J^3{}_S(y_1) + J^3{}_S(y_1)\right)\right\}$$
$$= \min\{J^3{}_A(x_1) + J^3{}_S(x_1) - J^3{}_A(x_1)J^3{}_S(x_1), J^3{}_A(y_1) + J^3{}_S(y_1) - J^3{}_A(y_1)J^3{}_S(y_1)\}$$
$$\leq J^3{}_A(a) + J^3{}_S(a) - J^3{}_A(a)J^3{}_S(a) = J^3{}_{A+S}(a).$$

Moreover, $K^3{}_A(x_1), K^3{}_S(y_1) < 1$, which implies that
$1 > \max\{K^3{}_A(x_1)K^3{}_S(x_1),\ K^3{}_A(y_1)K^3{}_S(y_1)\} \geq K^3{}_A(a)K^3{}_S(a) = K^3{}_{A+S}(a)$.
Thus $a \in (A + S)^*$.Then $A^* + S^* \subseteq (A + S)^*$.
Therefore, we have $(A + S)^* = A^* + S^*$.

(ii) Since $A \cap S = \chi_0{}^{FF}$, we need to show that $(A + S)_* = A_* + S_*$.
Suppose that $a \in (A + S)_*$. By definition, we have $J^3{}_{A+S}(a) = 1$.
This implies that $1 = J^3{}_A(a) + J^3{}_S(a) - J^3{}_A(a)J^3{}_S(a) = J^3{}_A(a)\left(1 - J^3{}_S(a)\right) + J^3{}_S(a)$
which means that $J^3{}_A(a) = 1$ or $J^3{}_S(a) = 1$.
Moreover, $0 = K^3{}_{A+S}(a) = K^3{}_A(a)K^3{}_S(a)$,
which implies that $J^3{}_A(a) = 0$ or $J^3{}_S(a) = 0$.
Thus $a \in A_*$ or $a \in S_*$, so that $a \in A_* + S_*$ and $(A + S)_* \subseteq A_* + S_*$.
Now, suppose that $a = x_1 + y_1 \in A_* + S_*$, where $x_1 \in A_*$ and $y_1 \in S_*$.
By definition, $J^3{}_A(x_1), J^3{}_S(y_1) = 1$. Thus,

$$1 = \min\{J^3{}_A(x_1) + J^3{}_S(x_1) - J^3{}_A(x_1)J^3{}_S(x_1), J^3{}_A(y_1) + J^3{}_S(y_1) - J^3{}_A(y_1)J^3{}_S(y_1)\}$$
$$\leq J^3{}_A(a) + J^3{}_S(a) - J^3{}_A(a)\, J^3{}_S(a) = J^3{}_{A+S}(a)$$

Moreover, $K^3{}_A(x_1), K^3{}_S(y_1) \leq 0$, which implies that
$0 = \max\{K^3{}_A(x_1)K^3{}_S(x_1),\ K^3{}_A(y_1)K^3{}_S(y_1)\} \geq K^3{}_A(a)K^3{}_S(a) = K^3{}_{A+S}(a)$.
Thus $a \in (A + S)_*$.Then $A_* + S_* \subseteq (A + S)_*$
Therefore, we have $(A + S)_* = A_* + S_*$.

IV. HOMOMORPHISM OF MODULES

Definition-4.1: Let A, S be two R-modules and $U \leq_{FF} A$ and $V \leq_{FF} S$. Consider an R-homomorphism $\varphi: A \rightarrow S$. For $s \in S$, we define

$$J_{\varphi(U)}(s)=\begin{cases}\max\{J_U(a): s=\varphi(a)\}, & \text{if } s\in Im(\varphi)\\ 0, & \text{otherwise}\end{cases}$$

and $K_{\varphi(U)}(s)=\begin{cases}\min\{J_U(a): s=\varphi(a)\}, & \text{if } s\in Im(\varphi)\\ 0, & \text{otherwise}\end{cases}$.

Now, we are ready to prove the following:

Theorem-4.2: Let $\varphi: A\to S$ be a homomorphism of modules. If 'D' is a fermatean fuzzy small sub-module of A, then $\varphi(D)$ is a fermatean fuzzy sub-module of S.

Proof:

Suppose that $\varphi(D)+U=\chi_S{}^{FF}$.Our aim is to prove that $U=\chi_S{}^{FF}$.

Let $s\in S$, then $1=J^3{}_{\varphi(D)+U}(s)=J^3{}_{\varphi(D)}(s)+J^3{}_U(s)-J^3{}_{\varphi(D)}(s)J^3{}_U(s)$.

In this case that $s\notin Im(\varphi)$, we obtain

$1=J^3{}_{\varphi(D)}(s)+J^3{}_U(s)-J^3{}_{\varphi(D)}(s)J^3{}_U(s)=J^3{}_U(s)$ and so $1=J^3{}_U(s)$ and $K^3{}_U(s)=0$.

If $s\in Im(\varphi)$, we have

$$\begin{aligned}1&=J^3{}_{\varphi(D)}(s)+J^3{}_U(s)-J^3{}_{\varphi(D)}(s)J^3{}_U(s)\\&=\max\{J^3{}_D(a)/\varphi(a)=s\}+J^3{}_U(s)-\max\{J^3{}_D(a)/\varphi(a)=s\}\,J^3{}_U(s)\\&=J^3{}_D(a)+J^3{}_U(s)-J^3{}_D(a)J^3{}_U(s), \text{ for some a in which } \varphi(a)=s.\\&=J^3{}_D(a)\left(1-J^3{}_U(s)\right)+J^3{}_U(s).\end{aligned}$$

If $J^3{}_D(a)=1$, then $D=\chi_A$ and this is a contradiction with the fact that D is a fermatean fuzzy small sub-module of A.

Thus $J^3{}_U(s)=1$ and $D=\chi_S$.

Moreover, $0=K^3{}_{\varphi(D)+U}(s)=K^3{}_{\varphi(D)}(s)K^3{}_U(s)$

$=K^3{}_D(a)K^3{}_D(s)$, for some a in which $\varphi(a)=s$.

Note that φ is one to one and so 'a' is unique.

By hypothesis $K^3{}_D(a)\neq 0$, so that $K^3{}_U(a)=0$.

Hence $D=\chi_S{}^{FF}$.

Note-4.3: (i) If φ is not one to one, then the above theorem need not be true. For instance, take S is the zero module and φ the zero homomorphism.

(ii) The converse of the above theorem need not be true. That is $\varphi: A\to S$ is a monomorphism of modules D is a fermatean fuzzy sub-module of A and $\varphi(D)$ is a fermatean fuzzy small sub-module of S, then it is not true in general that D is a fermatean fuzzy sub-module of A. For example, Let A be a fermatean fuzzy small sub-module of S and consider the inclusion $A\to S$. Then $\varphi(A)=A$ is a fermatean fuzzy sub-module of S but A is not fermatean fuzzy sub-module of A.

V. CONCLUSION

In this article, we introduced the notion of a fermatean fuzzy sub-module, contributing to the broader study of fuzzy algebraic structures. We discussed the fundamental concepts underlying fermatean fuzzy sub-modules and investigated several important results that help to clarify their properties and behavior within module theory. Moreover, we examined the relationship between small sub-modules and fermatean fuzzy sub-modules, highlighting key similarities and differences that deepen the understanding of how fuzzy structures can interact with classical algebraic concepts. In addition to these theoretical developments, we explored homomorphisms between fermatean fuzzy modules, analyzing how these mappings preserve or transform fuzzy module structures and offering insights into their potential applications. The findings presented in this article not only enrich the theoretical landscape of fuzzy algebra but also suggest possible directions for future research, particularly in exploring practical applications of fermatean fuzzy modules in areas such as decision-making, information systems, and computational intelligence.

VI. FUTURE DIRECTION

This work can be further extended to the framework of generalized orthopair fuzzy sets, offering a broader perspective for theoretical development. Additionally, the concepts and results presented here hold potential for application in solving multi-criteria decision-making problems, where managing uncertainty and complex preferences is essential.

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